



This is “Time Value of Money: Multiple Flows”, chapter 7 from the book [Finance for Managers \(index.html\)](#) (v. 0.1).

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Chapter 7

Time Value of Money: Multiple Flows

The Cash Comes In, The Cash Goes Out...

PLEASE NOTE: This book is currently in draft form; material is not final.

Mortgages, student loans, credit cards... sadly, most people's understanding of personal finance extends only so far as "pay what the bill says to pay". How do we know if we're paying the proper amount? And what would the effect be if I paid \$5 more than the minimum this month? The basic tools necessary to answer these questions have already been provided in the previous chapter, and now we'll explore how to use them to solve these very common problems.

7.1 Multiple Cash Flows

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LEARNING OBJECTIVES

1. Calculate the NPV of an investment.
2. Evaluate investment choices using NPV.

Jamie was up late watching infomercials, and saw that she could purchase the latest wonder product for either \$43, or three easy monthly installments (the first payment is immediate) of \$15. If she earns 4% APR compounded monthly in her savings account, which option should she pick?

The trick to comparing cash flows usually is to PV them all to the same point in time. When they have been discounted to the same time, then they can be added, subtracted, or compared without worry. Discounting all of the cash flows for an investment to the present, adding inflows and subtracting outflows, is called finding the **net present value (NPV)**¹. When deciding among investments, we typically wish to choose the one with the highest (or, in some cases, least negative) NPV.

The NPV of paying \$43 immediately is obviously $-\$43$. To find the NPV of the installment plan, we should look at the timeline.

Figure 7.1 NPV Example Timeline

4% APR compounded monthly gives a monthly interest rate of 0.3333%. (More discussion of example)

1. Discounting all of the cash flows for an investment to the present, adding inflows and subtracting outflows.

Many financial calculators have an NPV solving functionality to handle multiple cash flows. Please check the appendix for differences between the models, but the standard process is:

- Step 1: Enter Cash Flow worksheet.
- Step 1a: If entering new data (as opposed to editing what is already entered), clear the worksheet.
- Step 2: Enter CF0 (Net cash flows at time 0).
- Step 3a: Advance worksheet to enter C01 (Cash flow at time 1).
- Step 3b: Advance worksheet to enter F01 (The number of times C01 repeats).
- Repeat steps 3a and 3b until all are entered.
- Step 4: Press NPV and enter periodic interest rate.
- Step 5: Compute NPV.

In our example problem, the initial cash flow (CF0) is -\$15. Both of the next two cash flows are -\$15, so we can either enter them separately (C01: -15; F01: 1; C02: -15; F02: 1) or tell the calculator to repeat the entry (C01: -15; F01: 2). Both methods will give the proper answer. Here are the precise keystrokes for the latter method:

```
<CF> <2ND> <CLR WORK> -15 <ENTER> <DOWN ARROW> -15 <ENTER> <DOWN ARROW> 2 <ENTER> <NPV> 0.3333 <ENTER> <DOWN ARROW> <CPT>
```

Remember that interest rate should be entered as a percentage (0.3333 in this case, not 0.003333).

The corresponding spreadsheet function is:

=NPV(periodic rate, cash flows from period 1 to n) + net of initial cash flows

Note that the spreadsheet function expects cash flows to start in the first period: any initial cash flows need to be netted and added to the result. Like the financial calculator, positive values should be used for cash inflows, and negative values for outflows.

Figure 7.2 NPV in a Spreadsheet

KEY TAKEAWAYS

- The net present value (NPV) of an investment can be found by summing the PVs of all of its cash flows.
- Comparing investments is most easily done by comparing their NPVs: the higher the NPV, the better the investment.

EXERCISES

1. An investment costs \$1,000 today, but will produce cash flows of \$400 in year 1, \$500 in year 2, and \$600 in year 3. If interest rates are 10%, what is the NPV of this investment?
2. Investment A costs \$2,000 today, with cash inflows of \$400, \$400, \$400, and \$1,200 in years 1–4 respectively. Investment B costs \$1,000 today and will have one cash inflow of \$1,100 in 1 year. If interest rates are 8%, what are the NPVs of the investments? At this interest rate, which is the better investment? If interest rates are 5%, what are the NPVs? At this rate, which is the better investment?

7.2 Perpetuities

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LEARNING OBJECTIVES

1. Define a perpetuity.
2. Calculate the PV of a perpetuity.
3. Explain how the PV of a perpetuity is derived.

As discussed in the previous section, finding the NPV for a fixed set of cash flows will always work. But what if the cash flows were to continue forever? If the payments are constant over time, we can intuit what the value must be for the entire set, instead of figuring out each one separately: if I put \$200 into a bank account that promises me 5% per year, and I just withdraw the interest each year without ever touching the principal, how much will I get each year? The first year, the interest will be 5% of \$200, or \$10. Since I withdraw the interest, my second year balance is also \$200, so my interest is again \$10. In fact, I (or my heirs) can withdraw \$10 a year *forever* as long as I keep the original \$200 in the bank.

Figure 7.3 *Perpetuity Timeline*

Note that this is essentially the same as the bank offering the following: pay \$200 today, and the bank will pay the customer \$10/year. Therefore, at 5% interest, \$10/year forever must be the equivalent of exactly \$200 today! This arrangement, with constant periodic payments lasting forever, is called a **perpetuity**². The periodic payment must be the interest rate times the present value, since it is analogous to our bank deposit example above.

Equation 7.1 *Perpetuity Equation*

$$PV \times \text{interest rate} = \text{payment}$$

$$PV \times r = PMT$$

2. Constant periodic payments lasting forever.

$$PV = \frac{PMT}{r}$$

Note that the interest rate and the payment must be quoted in the same period; if the payments are monthly, then the monthly rate should be used. Also, the first payment is assumed to be at the end of the year; if the first payment was immediate, then the value would be higher by exactly the amount of one payment.

The fact that a stream of payments lasting forever has a specific finite value today is a surprising result for some people. The key is to realize that, because of the time value of money, each successive payment contributes a smaller and smaller amount to the PV.

Figure 7.4 Contribution of Payments over Time to Value of Perpetuity

Returning to the bank deposit example: if I don't withdraw the full 5% of the \$200, instead only withdrawing 4% (that is, 1% less), then I won't receive as much money this year (\$8 vs. \$10), but my principal balance will grow by 1% to \$202. If I continue to withdraw only 4%, my second year's withdraw will be $\$202 \times 4\% = \8.08 , which happens to be exactly 1% higher than my first year's withdrawal. Withdrawing 1% less causes both my principal to grow by 1% at the end of the year, and my following year's payment to grow by the same 1%.

Figure 7.5 Perpetuity with Growth Timeline

If I consistently withdraw less than the interest rate (the difference, interest rate minus withdrawal rate, being the growth rate, g), I can model the relationship using the following modified perpetuity equation:

Equation 7.2 Perpetuity with Constant Growth

PV now \times withdrawal rate = payment at end of year

PV now \times (interest rate - growth rate) = payment at end of year

$$PV_n \times (r - g) = PMT_{n+1}$$

$$PV_n = \frac{PMT_{n+1}}{(r-g)} \left\{ \text{for } g < r \right\}$$

Note that we are assuming that *some* money is withdrawn in each year, meaning that g must be less than r . And, when g equals zero, this equation reduces to our earlier equation.

What Do These Subscripts Mean?

Since our payments are changing over time, we use subscripts (the little “n” and “n+1” in the above equation) to differentiate one payment from another. Almost always, subscripts will denote time periods for us, such that today is 0, one time period in the future is 1, and so on. Unless we specifically say otherwise, the default time period will be years, so PMT_0 is the payment we just received today, and PMT_1 is what we expect to receive in exactly one year. If we want to be more generic and describe any given year, then PMT_n is year “n” and PMT_{n+1} is the year immediately following year “n”.

Since we know that our growth rate, g , is constant, given any payment we can find the following year’s payment by increasing it by g .

Equation 7.3 Payment Growth

payment at end of year = payment now + ($g \times$ payment now)

$$PMT_{n+1} = PMT_n + (g \times PMT_n) = PMT_n(1 + g)$$

$$PV_n = \frac{PMT_{n+1}}{(r-g)} = \frac{PMT_n(1+g)}{(r-g)} \quad \left\{ \text{for } g < r \right\}$$

Here PMT_n is the payment just paid, so technically its value isn’t included in the PV; we are only including it since we can use it to find the payment due next year.

KEY TAKEAWAYS

- A perpetuity is a constant stream of payments lasting forever, but a finite PV can be calculated.
- The perpetuity equation can be extended to account for cases where the payment grows over time, provided that the rate of growth is less than the interest rate.

EXERCISES

1. What is the present value of \$50 per year forever, if the current interest rate is 10%? What if only the first payment in a year's time is \$50, but then the payments grow at 2% per year?
2. I buy a perpetuity from a bank today for \$200. In five years, after collecting each year's payment, I then offer to sell the perpetuity back to the bank. If interest rates haven't changed, what should the perpetuity be worth at that time? If interest rates have risen, should the value of the perpetuity rise or fall?

7.3 Annuities

PLEASE NOTE: This book is currently in draft form; material is not final.

LEARNING OBJECTIVES

1. For an annuity, calculate the fifth variable given four of: PV, FV, rate, PMT, and n .
2. Calculate the PV of an annuity due.

Since, in practice, very few financial arrangements include “forever” as a reasonable timeframe, we should consider the value of cash flows over shorter periods. Any set of constant periodic payments lasting for a fixed amount of time is called an **annuity**³. Paying down a credit card, mortgages, lottery payouts, and more can all be modeled as annuities. All annuities can be valued using the “long way”, by calculating the NPV of the cash flows, but we can also value them using mathematical intuition.

Figure 7.6 Annuity Timeline (General)

Figure 7.7 Annuity Example Timeline

If we want to figure out the value of \$10/year for 15 years, and the current interest rate is 5%, then we can use our perpetuity formula to determine the value of the annuity in the following way:

1. Determine the value of an equivalent perpetuity (in this case, \$10/year at 5% gives a PV for the perpetuity of \$200).
2. Determine the PV of the value of the perpetuity at the end of the annuity period (in this case, the PV of \$200 in 15 years is \$96.20).
3. Subtract #2 from #1 to get the value of the annuity ($\$200 - \$96.20 = \$103.80$).

3. Constant periodic payments over a fixed amount of time.

Equation 7.4 PV of an Annuity

PV of annuity = PV of equivalent perpetuity – PV of perpetuity at end of annuity

$$PV = \frac{PMT}{r} - \frac{\frac{PMT}{r}}{(1+r)^n}$$

$$PV = \frac{PMT}{r} \times \left(1 - \frac{1}{(1+r)^n}\right)$$

Sometimes an annuity will also include an additional one-time cash flow at the conclusion of the annuity. In this case, it is a simple matter of adding the PV of this cash flow to your calculated value to get the total value of the annuity. Continuing our example, if the annuity delivered \$100 at its conclusion, then the PV would increase by the PV of \$100 in 15 years at 5%, or \$48.10, making the total value \$103.80 + \$48.10 = \$151.90.

Watch the Direction of Cash Flows

As we get more complex with our calculations, it becomes all the more important to keep track of whether our cash flows are inflows or outflows. For example, if a one-time payment in this example were instead a cash outflow, then our value would drop by \$48.10 to a total value of \$55.70.

Financial calculators and spreadsheets can also easily solve annuities. On the calculator, the PMT key accepts the input for the periodic payment, and any four inputs can be used to calculate the fifth.

Using a financial calculator:

<CLR TVM> 15 <N> 5 <I> 10 <PMT> 100 <FV> <CPT> <PV> should show the proper solution of -151.90.

Spreadsheet:

=PV(rate, nper, pmt, fv)

=PV(5%, 15, 10, 100)

-151.90

Also, PMT can be solved for given the other four inputs. =PMT(rate, nper, pv, fv) is the corresponding spreadsheet function. It is critical that we always input our rate, number of periods, and payment based upon the same time period. For example, if payments are monthly, then rate should be a monthly rate, and number of periods will be the number of months.

Why Is the Result Negative?

Calculators and spreadsheets always assume that a financial transaction is occurring, so they figure out the “fair value” based on the inputs. Since the NPV of the annuity’s inflows is \$151.90, the calculated answer is that you should be willing to pay \$151.90 (an outflow) to receive the scheduled payments in a completely “fair value” transaction.

With most annuities, each payment is assumed to occur at the end of each period. When payment in each period occurs at the beginning instead of the end, the annuity is called an **annuity due**⁴.

Figure 7.8 Annuity Due Timeline

While technically each payment is occurring at the beginning of its period instead of the end, in effect, the only change is that the very last payment has been moved to the very beginning! Thus, the value of an annuity due is higher by the difference between the PV of a payment at the end and the PV of a payment now (which, by definition, is equal to the total payment).

Equation 7.5 PV of an Annuity Due

PV of annuity due = PV of annuity + (payment - PV of payment at end)

$$PV_{due} = PV_{ordinary} + \left(PMT - \frac{PMT}{(1+r)^n} \right)$$

If our example were an annuity due, its value would increase by \$10 minus the PV of \$10 in 15 years at 5%, or $(\$10 - \$4.81) = \$5.19$.

4. An annuity with payments due at the beginning of each period instead of at the end.

Financial calculators can be set to solve for annuity due (see appendix). Most spreadsheet functions can be set to automatically calculate an annuity due by taking a fifth argument, “type”. When type is omitted, it is assumed to be calculating an ordinary annuity, but by using an input of “1” instead, an annuity due is calculated:

=PV(rate, nper, pmt, fv, type)

=PV(5%, 15, 10, 100, 1)

-157.09

KEY TAKEAWAYS

- Annuities are a fixed stream of payments over a finite time. Given four of PV, FV, rate, PMT, and n , the fifth variable can be solved for.
- Rate, PMT, and n should all be in the same time units!
- An annuity due is a type of annuity where the payments are due at the beginning of the period.

EXERCISE

1. Jill purchases an annuity that pays \$1,000 per month for the next 300 months, with no value at the end of the 300 months ($FV=0$). If interest rates are 12% APR, what is the fair PV of this annuity? What would it be worth if it were an annuity due?

7.4 Loan Amortization

PLEASE NOTE: This book is currently in draft form; material is not final.

LEARNING OBJECTIVES

1. Explain loan amortization.
2. Create an amortization schedule.

Fred owes \$1,500 on his credit card! When his monthly bill arrives, Fred is relieved to see that the minimum payment is only \$30. If the advertised rate is 20% APR (assume monthly compounding), how many months will it take Fred to pay off his debt?

If Fred makes his payments every month, the debt will finally be fully paid after 109 months, or over 9 years! We can verify this quickly using a financial calculator or spreadsheet.

=NPER(20%/12,30,-1500)

=108.4

During this time, the total amount of money paid by Fred will be about $109 * \$30 = \$3,270$, or over twice the original amount owed. This means that Fred paid more in interest ($\$3,270 - \$1,500 = \$1,770$ in interest) than he originally borrowed ($\$1,500$). On the plus side, Fred was able to slowly pay down the balance owed, so that he never had to pay more than \$30 in any month, but the total principal was paid by the end of the loan. This process of spreading out principal payments of a loan over time is called **amortization**⁵ (from the Latin root “mort-”, meaning death, as the loan balance slowly “dies” as the principal is paid down). Mortgages are a common example of this type of loan (also coming from the same Latin root), as the principal is paid down over 15, 30 or more years. A loan that pays only the interest payments, with no principal payments, is called an interest only loan. If the periodic payments are so low they don’t even cover the interest on the initial principal, then the loan will be a negative amortization loan, and the extra interest will be added to the principal due over time.

5. The process of spreading out the principal payments of a loan over time.

Figure 7.9 *Amortization Schedule for Credit Card Payments*

As each month passes, the outstanding principal balance decreases, so the interest due also decreases. Thus, over time, the amount paid toward principal grows from a small portion of the monthly payment to a larger portion. This is particularly important when evaluating mortgages, as often the interest portion is tax deductible, but the principal portion is not. If we were to continue the chart for the full 109 months, we would see that the last month would only require a partial payment (that is, less than \$30) to pay off the remaining principal.

Figure 7.10 *Principal and Interest Portion of Payments Over Time*

Of course, not all loans have constant payment schedules. Some loans have “teaser” periods, with low interest rates for a certain period of time. These might seem like fantastic deals, until we realize that the loan is actually negatively amortizing over the period, causing higher rates (or a longer payoff time) once the period is over.

KEY TAKEAWAYS

- Payments can be broken into principal and interest components.
- As principal is paid down, interest decreases. Future payments will, over time, accelerate in their rate of paying down principal.
- Negative amortization can actually increase principal balances over time.

EXERCISES

1. If Fred could afford to pay \$5 more than the minimum payment, how many months would it take to pay off the balance?
2. A 30 year fixed rate mortgage has monthly payments (and no payment due at the end). If the interest rate is 6% APR and the amount borrowed is \$200,000, what is the monthly payment due? Create an amortization schedule for this mortgage.

7.5 End-of-Chapter Assessment

End-of-Chapter Assessment Head

First paragraph.

Paragraph.

Paragraph.

Last paragraph.

End-of-Chapter Assessment Head

- 1.
- 2.
- 3.
- 4.
- 5.

End-of-Chapter Assessment Head

1.

- a.
- b.
- c.
- d.
- e.

2.

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- b.
- c.
- d.
- e.

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